Group Invariant Solutions without Transversality
and the Principle of Symmetric Criticality

Ian M. Anderson
Department of Mathematics
Utah State University
Logan, Utah 84322

Mark E. Fels
Department of Mathematics
Utah State University
Logan, Utah 84322

Charles G. Torre
Department of Physics
Utah State University
Logan, Utah 84322

Abstract.

We extend Lie's classical method for finding group invariant solutions to the case of non-transverse group actions. For this extension of Lie's method we identify a local obstruction to the principle of symmetric criticality. Two examples of non-transverse symmetry reductions for the potential form of Maxwell's equations are then examined.

KeyWords. Group invariant solutions, symmetric criticality.

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Let $\Delta = 0$ be a system of differential equations and let $\tilde{\Delta} = 0$ be a second system which is related to the first by a geometric transformation. For example, the second system could be derived from the first by the process of group reduction, a Backlund transformation, or a differential substitution. The question then arises whether formal geometric properties of the second system $\tilde{\Delta} = 0$ can be inferred from those of the original. For example, if $\Delta = 0$ is a system of Euler-Lagrange equations is $\tilde{\Delta} = 0$ also a system of Euler-Lagrange equations? To answer such questions a rigorous mathematical description of the process relating the two systems of equations is needed. In this article we will address this issue in the case where $\tilde{\Delta} = 0$ are the reduced equations for the group invariant solutions to $\Delta = 0$.

Lie's method of symmetry reduction for finding the group invariant solutions to partial differential equations is well-known (see, for example, Bluman and Kumi [5], Olver [8], Winternitz [11],[12]). However, these references make the hypothesis of *transversality* of the group action, an assumption which is not valid for most problems in field theory and differential geometry. In the next section we show how one can dispense with this assumption and find the reduced differential equations for group actions which are not transverse. In section 3 we extend the results in the article [1] and find another local obstruction to the principle of symmetric criticality [10] for non-transverse group actions. Finally in section 4 we find the reduced equations for two non-transverse symmetry groups of Maxwell's equations and we check the principle of symmetric criticality on these.

1. Group Invariant Solutions Without Transversality

1.1 Preliminaries.

Let M be an n-dimensional manifold and $\pi \colon E \to M$ a bundle over M. For this article it is sufficient to consider E as a trivial bundle $E = U \times \mathbf{R}^m$ where U is an open set in $\mathbf{R}^n$. The manifold M will serve as the space of independent variables and the bundle E plays the role of the total space of independent and dependent variables. Points of M will be labeled with local coordinates (x^i) and points of E with local coordinates (x^i, u^α). In terms of these coordinates the projection map π is given by $\pi(x^i, u^\alpha) = (x^i)$. We let $E_x = \pi^{-1}(x)$ for $x \in M$.

Let G be a finite dimensional Lie group which acts smoothly and projectably on E. That is, the action of each element of G is a fiber preserving transformation on E and, consequently, there is a

smooth induced action of G on M such that the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\ g\ } & E \\
\pi \downarrow & & \downarrow \pi \\
M & \xrightarrow{\ g\ } & M
\end{array}
$$

commutes. For projectable group actions the action of G on the space of sections of E is given by

$$
(g \cdot s)(x) = g \cdot s(g^{-1} \cdot x),
$$

where $s : M \to E$ is a smooth section.

Definition 1.1. *Let G be a smooth projectable group action on the bundle $\pi \colon E \to M$ and let $U \subset M$ be open. Then a smooth section $s \colon U \to E$ is G **invariant**, if for all $x \in U$ and $g \in G$ such that $g \cdot x \in U$,*

$$
s(g \cdot x) = g \cdot s(x). \tag{1.1}
$$

Let Γ be the Lie algebra of infinitesimal generators for the action of G on E. Since the action of G on E is assumed projectable any basis V_a, $a = 1, \ldots, p$, for Γ assumes the local coordinate form

$$
V_a = \xi_a^i(x)\partial_{x^i} + \eta_a^\alpha(x, u)\partial_{u^\alpha} \ . \tag{1.2}
$$

Suppose that $v_a \in \mathfrak{g}$ (the Lie algebra of G), and that V_a is the corresponding infinitesimal generator. Let s be an invariant section given in local coordinates by $s(x) = (x^i, s^\alpha(x^i))$. If we substitute $g = exp(tv_a)$ into equation (1.1), differentiate with respect to t and put $t = 0$ we obtain the *infinitesimal invariance equations*

$$
\xi_a^i(x)\frac{\partial s^\alpha}{\partial x^i} = \eta_a^\alpha(x, s^\alpha) \tag{1.3}
$$

for the invariant section s. If s is globally defined on all of M and if G is connected, then the infinitesimal invariance criterion (1.3) implies (1.1).

The condition that a group with infinitesimal generators V_a in (1.2) acts (infinitesimally) transversally is (see [5], [8]) .

$$
\mathrm{rank}[\xi_1(x), \ldots, \xi_p(x)] = \mathrm{rank}[\xi_1(x), \ldots, \xi_p(x), \eta_1(x, u), \ldots, \eta_p(x, u)] \ .
$$

Under the assumption of transversality the invariant section equation (1.3) is easily solved in terms of the infinitesimal invariants for Γ. In the next section we describe how to determine the invariant sections without this hypothesis.

1.2 Kinematic Reduction.

In order to study group invariant solutions to a differential equation we need to first characterize the G invariant sections of E and for this we make the following observation. Suppose that $p \in E$ and that there is a G invariant section $s : U \to E$ with $s(x) = p$, where $x \in U$. Let $G_x = \{ g \in G \mid g \cdot x = x \}$ be the **isotropy subgroup of G at x.** Then for every $g \in G_x$, we compute from (1.1)

$$g \cdot p = g \cdot s(x) = s(g \cdot x) = s(x) = p. \tag{1.4}$$

Consequently, if $g \in G_x$ and $p \in E_x$ and $g \cdot p \neq p$, then there can be **no** invariant section through the point p. In view of this, we define the **kinematic bundle $\kappa(E)$ for the action of G on E** by

$$\kappa(E) = \bigcup_{x \in M} \kappa_x(E),$$

where

$$\kappa_x(E) = \{ p \in E_x \mid g \cdot p = p \quad \text{for all} \quad g \in G_x \}.$$

It is easy to check that $\kappa(E)$ is a G invariant subset of E and therefore the action of G restricts to an action on $\kappa(E)$. The set $\kappa(E)$ is a generalization of the construction in [7].

Now construct the quotient spaces $\widetilde{M} = M/G$ and $\tilde{\kappa}(E) = \kappa(E)/G$ for the actions of G on M and $\kappa(E)$ and define the **kinematic reduction diagram for the action of G on E** to be the commutative diagram

$$
\begin{array}{ccccc}
\tilde{\kappa}(E) & \xleftarrow{\ \mathbf{q}_\kappa\ } & \kappa(E) & \xrightarrow{\ \iota\ } & E \\[2pt]
\tilde{\pi} \downarrow & & \pi \downarrow & & \downarrow \pi \\[2pt]
\widetilde{M} & \xleftarrow{\ \mathbf{q}_M\ } & M & \xrightarrow{\ id\ } & M.
\end{array}
\tag{1.5}
$$

In this diagram ι is the inclusion map of the kinematic bundle $\kappa(E)$ into E, the maps $\mathbf{q}_M$ and $\mathbf{q}_\kappa$ are the projection maps to the quotient spaces and $\tilde{\pi}$ is the surjective map induced by π.

We mention a few simple technical facts about these constructions.

Lemma 1.2.
i] *For all $p \in \kappa_x(E)$, $G_p \cong G_x$.*
ii] *If $\tilde{p} \in \tilde{\kappa}(E)$ and $x \in M$ satisfy $\tilde{\pi}(\tilde{p}) = \mathbf{q}_M(x)$, then there exists a unique point $p \in \kappa_x(E)$ such that $\mathbf{q}_\kappa(p) = \tilde{p}$.*

from ii] it can be immediately inferred that for every local section $\tilde{s} : \widetilde{U} \to \tilde{\kappa}(E)$ there exists a uniquely determined G invariant section $s : \mathbf{q}_M^{-1}(\widetilde{U}) \to \kappa(E)$ such that

$$\mathbf{q}_\kappa(s(x)) = \tilde{s}(\mathbf{q}_M(x)) . \tag{1.6}$$

The kinematic reduction diagram, along with (1.6), leads to the following existence theorem for invariant sections.

Theorem 1.3. *Suppose that E admits a kinematic reduction diagram such that i) $\kappa(E)$ is an embedded sub-bundle of E, ii) the quotient spaces $\widetilde{M}$ and $\tilde{\kappa}(E)$ are smooth manifolds, and iii) $\tilde{\pi} \colon \tilde{\kappa}(E) \to \widetilde{M}$ is a bundle. Let $\widetilde{U}$ be any open set in $\widetilde{M}$ and let $U = \mathbf{q}_M^{-1}(\widetilde{U})$. Then (1.6) defines a one-to-one correspondence between the G invariant smooth sections $s \colon U \to E$ and the smooth sections $\tilde{s} \colon \widetilde{U} \to \tilde{\kappa}(E)$.*

The proof of this theorem can be found in [2].

For generic group actions the transversality condition is replaced by the hypothesis that $\iota : \kappa(E) \to E$ is an embedding. When G is a compact Lie group acting by isometries on a Hermitian vector-bundle E this condition holds [6].

We now characterize infinitesimally invariant sections. If the rank of the coefficient matrix $[\xi_a^i(x)]$ is q, then there are locally defined functions $\phi_\epsilon^a(x)$, where $\epsilon = 1, \ldots, p - q$, such that $\sum_{a=1}^p \phi_\epsilon^a(x)\xi_a^i(x) = 0$. Consequently, using the infinitesimal invariant section equation (1.3), we find that

$$\sum_{a=1}^p \phi_\epsilon^a(x) \left(\xi_a^i(x)\frac{\partial s^\alpha}{\partial x^i} + \eta_a^\alpha(x, s^\beta(x)) \right) = \sum_{a=1}^p \phi_\epsilon^a(x^j)\eta_a^\alpha(x^j, s^\beta(x^j)) = 0 \tag{1.7}$$

are the algebraic equations constraining the invariant sections. Accordingly we define the ***infinitesimal kinematic bundle*** $\kappa^0(E) = \bigcup_{x \in M} \kappa_x^0(E)$ where

$$\kappa_x^0(E) = \{\, p = (x^j, u^\beta) \in E_x \mid \sum_{a=1}^p \phi_\epsilon^a(x^j)\eta_a^\alpha(x^j, u^\beta) = 0 \,\}$$

$$= \{\, p \in E_x \mid Z(p) = 0 \quad \text{for all} \quad Z \in \Gamma_x \,\}, \tag{1.8}$$

and $\Gamma_x = \{Z \in \Gamma | \pi_*(Z)(x) = 0\}$. It is easy to see by using infinitesimal methods that $\kappa(E) \subset \kappa^0(E)$ and if, for each $x \in M$, G_x is connected, then $\kappa(E) = \kappa^0(E)$.

The (infinitesimal) kinematic reduction diagram (1.5) in local coordinates takes the following form

$$
\begin{array}{ccccc}
(\tilde{x}^r, v^a) & \xleftarrow{\;\mathbf{q}_\kappa\;} & (\tilde{x}^r, \hat{x}^k, v^a) & \xrightarrow{\;\iota\;} & (\tilde{x}^r, \hat{x}^k, \iota^\alpha(\tilde{x}^r, \hat{x}^k, v^a)) \\
\tilde{\pi} \downarrow & & \pi \downarrow & & \pi \downarrow \\
(\tilde{x}^r) & \xleftarrow{\;\mathbf{q}_M\;} & (\tilde{x}^r, \hat{x}^k) & \xrightarrow{\;id\;} & (\tilde{x}^r, \hat{x}^k).
\end{array}
\tag{1.9}
$$

To derive this coordinate description of the kinematic reduction diagram we begin with the local coordinates $\tilde{\pi} \colon (\tilde{x}^r, v^a) \to (\tilde{x}^r)$ on the bundle $\tilde{\kappa}(E) \to \widetilde{M}$. Since $\mathbf{q}_M \colon M \to \widetilde{M}$ is a submersion, we can use the coordinates $\tilde{x}^r$ as part of a local coordinate system $(\tilde{x}^r, \hat{x}^k)$ on M where $k = 1..n - q$, and q is the dimension of the orbits of G on M. As a consequence of Lemma 1.1 ii] one can use $(\tilde{x}^r, \hat{x}^k, v^a)$ as a system of local coordinates on $\kappa(E)$. Let $(\tilde{x}^r, \hat{x}^k, u^\alpha) \to (\tilde{x}^r, \hat{x}^k)$ be a system of local

coordinates on E. Since $\kappa(E)$ is an embedded sub-bundle of E, the inclusion map $\iota\colon \kappa(E) \to E$ assumes the form

$$\iota(\tilde{x}^r, \hat{x}^k, v^a) = (\tilde{x}^r, \hat{x}^k, \iota^\alpha(\tilde{x}^r, \hat{x}^k, v^a)). \tag{1.10}$$

If $v^a = \tilde{s}^a(\tilde{x}^r)$ is a local section of $\tilde{\kappa}(E)$, then the corresponding G invariant section of E is given by

$$s^\alpha(\tilde{x}^r, \hat{x}^k) = \iota^\alpha(\tilde{x}^r, \hat{x}^k, \tilde{s}^a(\tilde{x}^r)). \tag{1.11}$$

This is the general local form of an invariant section without the transversality assumption which we will use in the examples.

1.3 Dynamic Reduction.

Let $\pi^k\colon J^k(E) \to M$ be the k-th order jet bundle of $\pi\colon E \to M$. A point $\sigma = j^k(s)(x)$ in $J^k(E)$ represents the values of a local section s and all its derivatives to order k at the point $x \in M$. Since G acts naturally on the space of sections of E by (1.3) the action of G on E prolongs to an action on $J^k(E)$ by setting

$$g \cdot \sigma = j^k(g \cdot s)(gx) \quad \text{where } \sigma = j^k(s)(g \cdot x) \ .$$

Now let $\pi\colon \mathcal{D} \to J^k(E)$ be a vector bundle over $J^k(E)$ and suppose that the Lie group G acts projectably on $\mathcal{D}$ in a manner which covers the action of G on $J^k(E)$. A **differential operator** is a section $\Delta\colon J^k(E) \to \mathcal{D}$. The differential operator Δ is G invariant if

$$g \cdot \Delta(\sigma) = \Delta(g \cdot \sigma)$$

for all $g \in G$ and all points $\sigma \in J^k(E)$. A section s of E defined on an open set $U \subset M$ is a solution to the differential equations $\Delta = 0$ if

$$\Delta(j^k(s)(x)) = 0 \quad \text{for all} \quad x \in U.$$

Often the action of G on $\mathcal{D}$ is naturally defined in terms of the action of G on E as in the case of Euler-Lagrange operators (see section 2.2).

We want to construct a bundle $\widetilde{\mathcal{D}} \to J^k(\tilde{\kappa}(E))$ and a differential operator $\tilde{\Delta}\colon J^k(\tilde{\kappa}(E)) \to \widetilde{\mathcal{D}}$ such that the correspondence (1.6) defines a 1-1 correspondence between the G invariant solutions of $\Delta = 0$ and the solutions of $\tilde{\Delta} = 0$. The required bundle $\widetilde{\mathcal{D}} \to J^k(\tilde{\kappa}(E))$ can not be constructed by a direct application of the kinematic reduction diagram to $\mathcal{D} \to J^k(E)$. This difficulty is circumvented by introducing the **bundle of invariant k- jets**, (Olver [8])

$$\text{Inv}^k(E) = \{\, \sigma \in J^k(E) \,|\, \sigma = j^k(s)(x_0),$$

$$\text{where} \quad s \quad \text{is a } G \text{ invariant section defined in a neighborhood of } x_0 \,\}.$$

The fundamental property we need of $\mathrm{Inv}^k(E)$ is that the quotient space $\mathrm{Inv}^k(E)/G$ coincides with the jet space $J^k(\tilde{\kappa}(E))$. We let $\mathcal{D}_{\mathrm{Inv}} \to \mathrm{Inv}^k(E)$ be the restriction of $\mathcal{D}$ to the bundle of invariant sections and to this bundle we now apply our reduction procedure to arrive at the **dynamic reduction diagram**

$$
\begin{array}{ccccccc}
\tilde{\kappa}(\mathcal{D}_{\mathrm{Inv}}) & \xleftarrow{\ \mathfrak{q}\ } & \kappa(\mathcal{D}_{\mathrm{Inv}}) & \xrightarrow{\ \iota\ } & \mathcal{D}_{\mathrm{Inv}} & \xrightarrow{\ \iota^k\ } & \mathcal{D} \\
\tilde{\pi}\downarrow & & \pi\downarrow & & \pi\downarrow & & \downarrow\pi \\
J^k(\tilde{\kappa}(E)) & \xleftarrow{\ \mathfrak{q}_{\mathrm{Inv}}\ } & \mathrm{Inv}^k(E) & \xrightarrow{\ Id\ } & \mathrm{Inv}^k(E) & \xrightarrow{\ \iota^k\ } & J^k(E).
\end{array}
\tag{1.12}
$$

Any G invariant differential operator $\Delta\colon J^k(E) \to \mathcal{D}$ restricts to a G invariant differential operator $\Delta\colon \mathrm{Inv}^k(E) \to \mathcal{D}_{\mathrm{Inv}}$ which determines a differential operator $\tilde{\Delta}\colon J^k(\tilde{\kappa}(E)) \to \tilde{\kappa}(\mathcal{D}_{\mathrm{Inv}})$. We call the differential operator the reduced operator and the equations $\tilde{\Delta} = 0$ the reduced equations. It is not difficult to prove the following theorem.

Theorem 1.4. *The solutions to the reduced differential equations $\tilde{\Delta} = 0$ are in one-to-one correspondence with the G invariant solutions for the original equations $\Delta = 0$.*

To describe diagram (1.12) in local coordinates, we begin with the coordinate description (1.9) of the kinematic reduction diagram and we let

$$
(\tilde{x}^r, \hat{x}^k, u^\alpha, u_r^\alpha, u_k^\alpha, u_{rs}^\alpha, u_{rk}^\alpha, u_{kl}^\alpha, \dots)
\tag{1.13}
$$

be coordinates on $J^k(E)$. Since the invariant sections are parameterized by functions $v^a = v^a(\tilde{x}^r)$ the coordinates for $\mathrm{Inv}^k(E)$ are

$$
(\tilde{x}^r, \hat{x}^k, v^a, v_r^a, v_{rs}^a, \dots)
$$

and the inclusion map

$$
\iota^k\colon \mathrm{Inv}^k(E) \to J^k(E)
$$

is given by

$$
\iota^k(\tilde{x}^r, \hat{x}^k, v^a, v_r^a, v_{rs}^a, \dots) = (\tilde{x}^r, \hat{x}^k, u^\alpha, u_r^\alpha, u_k^\alpha, u_{rs}^\alpha, u_{rk}^\alpha, u_{kl}^\alpha, \dots),
\tag{1.14}
$$

where the derivative terms are obtained by taking derivatives of $u^\alpha = \iota^\alpha(\tilde{x}^r, \hat{x}^k, v^a)$, for example,

$$
u_r^\alpha = \frac{\partial \iota^\alpha}{\partial \tilde{x}^r} + \frac{\partial \iota^\alpha}{\partial v^a} v_r^a, \qquad u_k^\alpha = \frac{\partial \iota^\alpha}{\partial \hat{x}^k}.
$$

The coordinates on $J^k(\tilde{\kappa}(E))$ are then

$$
(\tilde{x}^r, v^a, v_r^a, v_{rs}^a, \dots).
$$

Next let $\mathbf{f}^A$ be a local frame field for the vector bundle $\mathcal{D}$. The differential operator $\Delta\colon J^k(E) \to \mathcal{D}$ can be written in terms of the coordinates (1.13) on $J^k(E)$ and this local frame as

$$\Delta = \Delta_A(\tilde{x}^r, \hat{x}^k, u^\alpha, u_r^\alpha, u_k^\alpha, u_{rs}^\alpha, u_{rk}^\alpha, u_{kl}^\alpha \dots)\,\mathbf{f}^A. \tag{1.15}$$

The restriction of Δ to $\mathrm{Inv}^k(E)$ defines the section $\Delta_{\mathrm{Inv}}\colon \mathrm{Inv}^k(E) \to \mathcal{D}_{\mathrm{Inv}}$ by

$$\Delta_{\mathrm{Inv}} = \Delta_{\mathrm{Inv},A}(\tilde{x}^r, \hat{x}^k, v^a, v_r^a, v_{rs}^a, \dots)\,\mathbf{f}^A, \tag{1.16}$$

where the component functions are $\Delta_{\mathrm{Inv},A}(\tilde{x}^r, \hat{x}^k,, v^a, v_r^a, v_{rs}^a, \dots) = \Delta_A \circ \iota^k$. Since Δ is a G invariant differential operator, Δ_{Inv} is a G invariant differential operator and hence necessarily factors through the kinematic bundle $\kappa(\mathcal{D}_{\mathrm{Inv}})$, and is a section

$$\Delta_{\mathrm{Inv}}\colon \mathrm{Inv}^k(E) \to \kappa(\mathcal{D}_{\mathrm{Inv}})\ .$$

The existence theorem for invariant sections implies that we can also find a locally defined G invariant frame $\mathbf{f}_{\mathrm{Inv}}^Q$ for $\kappa(\mathcal{D}_{\mathrm{Inv}}) \to \mathrm{Inv}^k(E)$ in terms of which the invariant operator Δ_{Inv} can be expressed as

$$\Delta_{\mathrm{Inv}} = \Delta_{\mathrm{Inv},Q}(\tilde{x}^r, \hat{x}^k, v^a, v_r^a, v_{rs}^a, \dots)\,\mathbf{f}_{\mathrm{Inv}}^Q.$$

The reduced operator is now easily determined.

2. The Principle of Symmetric Criticality

2.1 Lagrangian Reduction.

Let $\lambda = L\nu$ be a G invariant k^{th} order Lagrangian, where L is a smooth function on $J^k(E)$ and ν is a volume form on M. For simplicity ν is assumed to be G invariant and L a differential invariant. In order to define the reduced Lagrangian $\tilde{\lambda}$ on $J^k(\tilde{\kappa}(E))$, we start with a G invariant q chain $\chi\colon M \to \wedge^q(TM)$ and define the equivariant bundle map

$$\rho_\chi\colon \wedge^*(T^*M) \to \wedge^{*-q}(T^*M),$$

by $\rho_\chi(\alpha) = \chi \lrcorner\, \alpha$. The map ρ_χ induces a map on G invariant sections

$$\rho_\chi\colon \Omega^*(M)^G \to \Omega^{*-q}(M)^G\ .$$

The image of a differential form under ρ_χ is a G basic form, from which we further obtain a map

$$\tilde{\rho}_\chi\colon \Omega^*(M)^G \to \Omega^{*-q}(\widetilde{M})\ .$$

For example, if $G = SO(3)$ is acting in the standard way on $M = \mathbf{R}^3 - (0,0,0)$, then ([1] pg. 615)

$$\chi = r\left(z\partial_x \wedge \partial_y - y\partial_x \wedge \partial_z + x\partial_y \wedge \partial_z\right), \tag{2.1}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, and

$$\rho_\chi(dx \wedge dy \wedge dz) = r(x\,dx + y\,dy + z\,dz)$$

so

$$\tilde{\rho}_\chi(dx \wedge dy \wedge dz) = r^2 dr \ .$$

Again see [1] or [4] for more details and examples.

Using the maps $\tilde{\rho}_\chi$ and $\iota^k : \text{Inv}^k(E) \to J^k(E)$ we define the reduced Lagrangian as

$$\tilde{\lambda} = \tilde{\rho}_\chi((\iota^k)^*\lambda)\,. \tag{2.2}$$

We point out that the role of the map $\tilde{\rho}_\chi$ is to reduce the number of independent variables in the volume form.

Let $Vert(E)$ be the vertical bundle for $\pi : E \to M$. Then from the maps $\pi_E^{2k} : J^{2k}(E) \to E$ and $\pi_M^{2k} : J^{2k}(E) \to M$ we construct the **bundle of source forms** $\mathcal{D} \to J^{2k}(E)$ by

$$\mathcal{D} = (\pi_E^{2k})^* \left(Vert^*(E)\right) \wedge (\pi_M^{2k})^* \left(\wedge^n T^* M\right)$$

where * denotes pullback. A point $\omega \in \mathcal{D}_\sigma$, $\sigma \in J^{2k}(E)$ is a differential form

$$\omega = A_\alpha\, du^\alpha \wedge \nu \ .$$

The Euler-Lagrange form $\mathcal{E}(\lambda)$ of a kth order Lagrangian is a section of $\mathcal{D}$ and may be written in local coordinates (x^i, u^α) as

$$\mathcal{E}(\lambda) = E_\alpha(L) du^\alpha \wedge \nu \ ,$$

where $E_\alpha(L)$ are the Euler Lagrange expressions for $\lambda = L\nu$.

At this point there are two ways to proceed in the reduction of the Euler-Lagrange equations. First, given a G invariant Lagrangian λ the Euler-Lagrange operator $\mathcal{E}(\lambda)$ is also G-invariant. Using the dynamic reduction diagram (1.12) in section 1.3 we can then compute the reduced operator $\widetilde{\mathcal{E}(\lambda)}$ whose solutions determine the group invariant solutions. The reduced operator $\widetilde{\mathcal{E}(\lambda)}$ is a section of $\tilde{\kappa}(\mathcal{D}_{\text{Inv}})$. Alternatively, if $\tilde{\lambda}$ is the reduced Lagrangian, then $\mathcal{E}(\tilde{\lambda})$ is a source form on $J^{2k}(\tilde{\kappa}(E))$ which, using the coordinates in (1.9)

$$\mathcal{E}(\tilde{\lambda}) = E_a(\tilde{L}) dv^a \wedge \tilde{\rho}_\chi(\nu)$$

where $\tilde{\rho}_\chi(\nu)$ is a volume form on $\widetilde{M}$. Therefore it is **not** possible to compare the sections $\widetilde{\mathcal{E}(\lambda)}$ and $\mathcal{E}(\tilde{\lambda})$ since they are sections of different bundles and consequently, the solutions to $\widetilde{\mathcal{E}(\lambda)} = 0$ and $\mathcal{E}(\tilde{\lambda}) = 0$ may be different. This leads to the following principle.

The Principle of Symmetric Criticality (PSC). *Let G be a Lie group acting projectably on E and suppose there exists a G invariant chain χ. Let λ be a fixed but arbitrary G invariant Lagrangian with $\tilde{\lambda}$ the reduced Lagrangian, and let $\widetilde{\mathcal{E}(\lambda)}$ be the reduced Euler-Lagrange operator. If, for every choice of λ the submanifolds $\mathcal{E}(\tilde{\lambda})^{-1}(0) \subset J(\tilde{\kappa}(E))$ and $\widetilde{\mathcal{E}(\lambda)}^{-1}(0) \subset J(\tilde{\kappa}(E))$ coincide, then we say the principle of symmetric criticality holds for the action of G on E.*

Note that if PSC holds then every locally defined solution $\tilde{s} : \widetilde{U} \to \tilde{\kappa}(E)$ to $\mathcal{E}(\tilde{\lambda}) = 0$ determines though (1.6) a solution s to $\mathcal{E}(\lambda) = 0$. Of course the principle can often fail, see $[1]$, $[10]$.

Our objective is to determine the local obstruction to this principal which arises for non-transverse group actions.

A local obstruction to the commutation relation

$$\mathcal{E}\left(\tilde{\rho}_\chi (\iota^k)^* \lambda\right) = \tilde{\rho}_\chi \mathcal{E}((\iota^k)^* \lambda). \tag{2.3}$$

was found in $[1]$ and is described in the following Theorem.

Theorem 2.1. *If $H^q(\Gamma_x, G_x) \neq 0$ for all $x \in M$, then there exists a locally defined map $\tilde{\rho}_\chi : \Omega^*(M)^G \to \Omega^{*-q}(\widetilde{M})$ such that (2.3) holds, and moreover $\tilde{\rho}_\chi$ is a co-chain map, i.e. $d\,\tilde{\rho}_\chi = \tilde{\rho}_\chi\, d$.*

For a description of the Lie algebra cohomology condition

$$H^q(\Gamma_x, G_x) \neq 0 \tag{2.4}$$

see $[1]$ pg. 650. We now find

Lemma 2.2. *Suppose that $H^q(\Gamma, G_x) \neq 0$ for all $x \in M$. Let $\widetilde{U} \subset \widetilde{M}$ and let $\tilde{s} : \widetilde{U} \to \tilde{\kappa}(E)$ be a solution to the Euler-Lagrange equations $\mathcal{E}(\tilde{\lambda}) = 0$ for the reduced Lagrangian. Then the corresponding invariant section $s : U \to E$ from (1.6) is a solution to $(\iota^{2k})^* \mathcal{E}(\lambda) = 0$.*

Proof. It follows from the Lemma 2.1 that

$$\mathcal{E}(\tilde{\lambda}) = \mathcal{E}\left(\rho_\chi (\iota^k)^* \lambda\right) = \tilde{\rho}_\chi \mathcal{E}((\iota^k)^* \lambda),$$

which together with the basic fact about the Euler-Lagrange operators $[8]$

$$\mathcal{E}((\iota^k)^* \lambda) = (\iota^{2k})^* \mathcal{E}(\lambda) \tag{2.5}$$

proves the lemma. ∎

Note however that this Lemma does **not** guarantee that the invariant section $s : U \to E$ is a solution to the original Euler-Lagrange equations $\mathcal{E}(\lambda) = 0$. Let's see why.

2.2 The Palais Condition.

In this section we will assume that the cohomology condition (2.4) in Theorem 2.1 holds so that (2.3) is valid.

The obstruction to PSC we have hinted at arises from differences in the two types of reductions which are occurring. First, if λ is a G invariant Lagrangian, then the Euler-Lagrange operator $\Delta = \mathcal{E}(\lambda)$ is G invariant, and the right side of the dynamic reduction diagram gives

$$\Delta_{\mathrm{Inv}} = E_\alpha(L)\big|_{\mathrm{Inv}} du^\alpha \wedge \nu \,,$$

and a solution $\tilde{s} : \widetilde{U} \to \tilde{\kappa}(E)$ to the reduced equations $\widetilde{\mathcal{E}(\lambda)} = 0$ defines through (1.6) an invariant solution $s : U \to E$ to $E_\alpha(L)\big|_{j(s)(x)} = 0$. On the other hand if we reduce the Lagrangian λ as in (2.2), we find, using the formula in (2.5)

$$\mathcal{E}\left((\iota^k)^*\lambda\right) = (\iota^{2k})^*\mathcal{E}(\lambda) = \left(\frac{\partial \iota^\alpha}{\partial v^a}E_\alpha(L)\right)\Bigg|_{\mathrm{Inv}} dv^a \wedge \nu \ .$$

Therefore if $\tilde{s} : \widetilde{U} \to \tilde{\kappa}(E)$ is a solution to the equations $\mathcal{E}(\tilde{\lambda}) = 0$ we have by Lemma 2.2 that the corresponding invariant section s in (1.6) is a solution to

$$\left(\frac{\partial \iota^\alpha}{\partial v^a}E_\alpha(L)\right)\Bigg|_{j(s)(x)} = 0 \ .$$

Clearly this does not guarantee that $E_\alpha(L)\big|_{j(s)(x)} = 0$, or that $\tilde{s}$ is a solution to the reduced equations.

Conditions under which solutions to the equations $\mathcal{E}(\tilde{\lambda}) = 0$ are solutions to $\widetilde{\mathcal{E}(\lambda)} = 0$ can be obtained by a detailed analysis of Δ_{Inv} and $(\iota^{2k})^*\mathcal{E}(\lambda)$.

Suppose $\Delta = \mathcal{E}(\lambda)$ is an invariant section of the bundle of source forms $\mathcal{D}$, and let $\sigma \in \mathrm{Inv}^{2k}(E)$ and $x_0 = \pi^M(\sigma) \in M$ and $Z \in \Gamma_{x_0}$. Then on account of the invariance of ν the Lie derivative of Δ is

$$(\mathcal{L}_Z\Delta)\big|_\sigma = \left(pr^{2k}Z\left(E_\alpha(L)\right) + E_\beta(L)\frac{\partial \eta_a^\beta}{\partial u^\alpha}\right)\Bigg|_\sigma du^\alpha \wedge \nu.$$

This leads to the isotropy condition

$$0 = \left(\Delta_\beta \frac{\partial \eta^\beta}{\partial u^\alpha}\right)\Bigg|_\sigma du^\alpha \wedge \nu \tag{2.6}$$

which determines the fibres of $\kappa^0(\mathcal{D}_{\mathrm{Inv}})$. A geometric description of (2.6) can be given in terms of the **vertical linear isotropy representation**. Suppose $\kappa(E) \subset E$ is an embedded sub-bundle and let $Vert(\kappa(E))$ be the restriction (or pullback) of the vertical bundle over E to $\kappa(E)$. Since G is fibre-preserving, the differential of the action of G on E induces an action of G on $Vert(E)$ which, because $\kappa(E) \subset E$ is a G invariant set, restricts to an action on $Vert(\kappa(E))$. We then make the following definition.

Definition 2.2. *Let $p \in \kappa(E)$, $g \in G_p$ and $Y_p \in Vert_p\kappa(E)$. Then the **vertical linear isotropy representation** on $\kappa(E)$, $\rho_p : G_p \to GL(Vert_p\kappa(E))$ is defined by*

$$\rho_p(g)Y_p = g_*Y_p . \tag{2.7}$$

Note by Lemma 1.1 i] this is also a representation of G_x, where $x = \pi(p)$.

The differential of ρ_p is a homomorphism $\rho_p : \Gamma_p \to gl(Vert_p\kappa(E))$ called the **infinitesimal vertical linear isotropy representation**. If $Z = \xi^i(x)\partial_{x^i} + \eta^\alpha(x,u)\partial_{u^\alpha} \in \Gamma_{(x_0,u_0)}$ then, in the standard basis ∂_{u^α}, we have

$$\rho_{(x_0,u_0)}(Z) = \left(\frac{\partial \eta^\beta}{\partial u^\alpha}\bigg|_{(x_0,u_0)} \right) .$$

Equation (2.6) now proves the following Lemma.

Lemma 2.4. *If $\sigma \in \mathrm{Inv}^{2k}(E)$ and $p = \pi_E(\sigma)$, then $\kappa_\sigma(\mathcal{D}_{\mathrm{Inv}}) \cong [Vert_p^*(\kappa(E))]^{G_x}$.*

The invariants $[Vert_p^*(\kappa(E))]^{G_x}$ are computed using the dual of the representation (2.7). It is not difficult to check

Lemma 2.5. *$\iota_*Vert_p(\kappa(E)) = [Vert_{\iota(p)}E]^{G_x}$.*

Define the subspace

$$\left([Vert_p(E)]^{G_x} \right)^\perp = \{\alpha \in Vert_p^*(E) \mid \alpha(X) = 0 \text{ for all } X \in [Vert_p(E)]^{G_x}\} .$$

The pullback construction for $\mathcal{D}$ and Lemma 2.5 allows us to identify $\left([Vert_p(E)]^{G_x} \right)^\perp$ with the subspace of $\mathcal{D}_{\mathrm{Inv},\sigma}$, where $\pi_E^{2k}(\sigma) = p$, given by

$$\left([Vert_p(E)]^{G_x} \right)^\perp \cong \{A_\alpha du^\alpha \wedge \nu \in \mathcal{D}_{\mathrm{Inv},\sigma} \mid A_\alpha \frac{\partial \iota^\alpha}{\partial v_a} = 0 \} ,$$

and the coordinates in (1.14) are being used. We now come to the key result.

Theorem 2.5. *Suppose that λ is a G invariant Lagrangian and that $H^q(\Gamma_x, G_x) \neq 0$ for all $x \in M$. If*

$$[Vert_p^*(E)]^{G_x} \cap \left([Vert_p(E)]^{G_x} \right)^\perp = 0 \quad \text{for all } p \in \kappa(E), \tag{2.8}$$

then every solution $\tilde{s} : \widetilde{M} \to \tilde{\kappa}(E)$ to $\mathcal{E}(\tilde{\lambda}) = 0$, where $\tilde{\lambda}$ is reduced Lagrangian in (2.2), defines by (1.6), a solution to $\mathcal{E}(\lambda) = 0$. In other words, the PSC holds.

We will prove this under the hypothesis that the isotropy groups are connected so that we may use infinitesimal methods.

Proof. Let $\tilde{s} : \widetilde{U} \to \tilde{\kappa}(E)$ be a solution to $\mathcal{E}(\tilde{\lambda}) = 0$ and let $s : U \to \kappa(E)$ be the corresponding invariant section from (1.6). Using the isomorphism in Lemma 2.4, we have

$$E_\alpha(L)\big|_{j(s)(x)} \in [Vert^*_{(x,s(x))}(E)]^{G_x} .$$

We also have, by Lemma 2.2, that s satisfies

$$\left(\frac{\partial \iota^\alpha}{\partial v^a} E_\alpha(L) \right)\bigg|_{j(s)(x)} = 0,$$

therefore by Lemma 2.5

$$E_\alpha(L)\big|_{j(s)(x)} \in \left([Vert_p(E)]^{G_x} \right)^\perp$$

where $p = (x, s(x))$. Thus, for each $x \in U$,

$$E_\alpha(L)\big|_{j(s)(x)} \in [Vert^*_{(x,s(x))}(E)]^{G_x} \cap \left([Vert_{(x,s(x))}(E)]^{G_x} \right)^\perp$$

which vanishes by (2.8) and s is an invariant solution to $E_\alpha(L) = 0$. ∎

It is also possible to prove that (2.8) is necessary for the PSC to hold. We call (2.8) *the Palais condition* due to its similarity to the condition given in Palais' original article [10] .

3. Example - Maxwell's Equations

The base manifold in this case is $M = \mathbf{R}^4$ with coordinates (t, x, y, z) and $E = T^*M$. The Minkowski metric is $\eta = diag(+, -, -, -)$ and we use (x^a), $a = 0, 1, 2, 3$, and $(x, y, z) = (x^i), i = 1, 2, 3$ when convenient. A section of E is a differential form $\alpha = u_a dx^a$. The form α is the potential in Maxwell's equations which are the Euler-Lagrange equations of the first-order Lagrangian

$$\lambda = |d\alpha|^2 \nu , \tag{3.1}$$

where $\nu = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ and $|d\alpha|^2 = \eta^{ab} \eta^{cd} u_{[a,c]} u_{[b,d]}$. The Euler-Lagrange equations are $\Delta = \Delta^a du_a \wedge \nu$, where

$$\Delta^0 = u_{0,xx} + u_{0,yy} + u_{0,zz} - u_{1,tx} - u_{2,ty} - u_{3,tz} ,$$
$$\Delta^1 = -u_{0,tx} + u_{1,tt} - u_{1,yy} - u_{1,zz} + u_{2,xy} + u_{3,xz} ,$$
$$\Delta^2 = -u_{0,ty} + u_{1,xy} + u_{2,tt} - u_{2,xx} - u_{2,zz} + u_{3,yz} ,$$
$$\Delta^3 = -u_{0,tz} + u_{1,xz} + u_{3,yz} + u_{3,tt} - u_{3,xx} - u_{3,yy}.$$

We now consider two non-transverse groups actions, find the reduced equations and investigate the principle of symmetric criticality.

Example 3.1. $\Gamma = \{\epsilon_{kij}x^i\partial_{x^j} - \epsilon_{kij}u_i\partial_{u_j}\}$

The Lie algebra Γ is obtained from the infinitesimal generators of $SO(3)$ acting in the standard way on E. At the point $\mathbf{x}_0 = (t_0, x_0^k)$ we have the isotropy vector-field

$$Z = x_0^k\left(\epsilon_{kij}x^i\partial_{x^j} - \epsilon_{kij}u_i\partial_{u_j}\right) \in \Gamma_{\mathbf{x}_0} . \tag{3.2}$$

The sub-bundle $\kappa(E) \subset E$ at the point $\mathbf{x}_0$ is given by the vanishing of $x_0^i\epsilon_{kij}u_i$ (the cross product of (x_0^i) and (u_i)). Therefore $u_0 = v$, $u_i = wx_0^i$, with v, w being fibre coordinates on $\kappa_{\mathbf{x}_0}(E)$. The inclusion $\kappa(E) \rightarrow E$ is given by

$$\iota^k(t, x, y, z\,;\, v, w) \rightarrow (t, x, y, z\,;\, v, wx, wy, wz) .$$

The functions t and $r = \sqrt{x^2 + y^2 + z^2}$ are invariants on M so that in local coordinates the kinematic reduction diagram is

$$
\begin{array}{ccccc}
(t, r\,;\, v, w) & \xleftarrow{\;\mathbf{q}_\kappa\;} & (t, x^i\,;\, v, w) & \xrightarrow{\;\;\iota\;\;} & (t, x^i\,;\, u_a) \\
\tilde{\pi}\downarrow & & \pi\downarrow & & \downarrow\pi \\
(t, r) & \xleftarrow{\;\mathbf{q}_M\;} & (t, x^i) & \xrightarrow[id]{} & (t, x^i).
\end{array}
\tag{3.3}
$$

The invariant sections are now easily determined to be

$$u_0 = v(t, r) , \quad u_i = w(t, r)x^i .$$

In order to determine the reduced equations we compute the fibres of $\mathcal{D}_{\text{Inv}}$ by using Lemma 2.4. At the point $p_0 = (t_0, x_0, y_0, z_0\,;\, v_0, w_0x_0, w_0y_0, w_0z_0) \in \kappa(E)$ the vector-field Z in (3.2) lies in Γ_{p_0}. The (infinitesimal) vertical linear isotropy representation in the coordinate basis ∂_{u_a} is

$$\rho_{p_0}(Z) = \frac{\partial}{\partial u_l}x_0^k\epsilon_{kij}u_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -z_0 & y_0 \\ 0 & z_0 & 0 & -x_0 \\ 0 & -y_0 & x_0 & 0 \end{pmatrix} .$$

The fibres of $\kappa(\mathcal{D}_{\text{Inv}})$ are two dimensional and the inclusion into $\mathcal{D}_{\text{Inv}}$ is easily found by computing the null space of the transpose of the matrix above.

The sections

$$S_0 = du_0 \wedge \nu \quad \text{and} \quad S_1 = \sum_{i=1}^{3} x^i du_i \wedge \nu$$

of $\mathcal{D}$ are invariant. The reduced operator is easily obtained from

$$\Delta_{\text{Inv}} = \left(v_{rr} + \frac{2}{r}v_r - rw_{tr} - 3w_t \right) S_0 + \left(w_{tt} - \frac{1}{r}v_{tr} \right) S_1 \ .$$

To compute the Palais condition (2.8) we have

$$[Vert_p \kappa(E)]^{G_x} = \begin{pmatrix} a \\ bx_0 \\ by_0 \\ bz_0 \end{pmatrix} \ , \quad [Vert_p^* \kappa(E)]^{G_x} = (a, bx_0, by_0, bz_0) \qquad a, b \in \mathbf{R}$$

and the Palais condition (2.8) is satisfied. The cohomology condition (2.4) is also easily checked, and the principle of symmetric criticality holds. (Note that this also follows from the symmetry group being compact.)

For completeness we compute the reduced Lagrangian. The Lagrangian in (3.1) restricted to $\text{Inv}^k(E)$ is

$$\lambda\big|_{\text{Inv}} = -\frac{1}{2}(rw_t - v_r)^2 \nu \ ,$$

and evaluating $\lambda\big|_{\text{Inv}}$ on the chain in (2.1) we obtain the reduced Lagrangian

$$\tilde{\lambda} = \lambda\big|_{\text{Inv}}(\chi) = \frac{1}{2}(rw_t - v_r)^2 r^2 dt \wedge dr \ .$$

The reduced equations agree with the Euler-Lagrange equations of the reduced Lagrangian.

Example 3.2. $\Gamma = \{V_1 = \partial_y, V_2 = \partial_t - \partial_z, V_3 = (t+z)\partial_y + y(\partial_t - \partial_z) + (u_3 - u_0)\partial_{u_2} - u_2(\partial_{u_0} + \partial_{u_3})\}$

At the point $\mathbf{x}_0 = (t_0, x_0^k)$ the isotropy vector-field is

$$Z = V_3 - (t_0 + z_0)V_1 - y_0 V_3 = (u_3 - u_0)\partial_{u_2} - u_2(\partial_{u_0} + \partial_{u_3}) \in \Gamma_{\mathbf{x}_0} \ . \qquad (3.4)$$

The sub-bundle $\kappa(E) \subset E$ at the point $\mathbf{x}_0$ is given by $u_3 = -u_0$, $u_2 = 0$ and the inclusion $\kappa(E) \to E$ is

$$\iota(t, x, y, z \,;\, v, w) \to (t, x, y, z \,;\, v, w, 0, v) \ .$$

The functions $r = t + z$ and x are invariants and the kinematic reduction diagram is

$$\begin{array}{ccccc}
(r, x \,;\, v, w) & \xleftarrow{\ \mathbf{q}_\kappa\ } & (t, x^i \,;\, v, w) & \xrightarrow{\ \iota\ } & (t, x^i \,;\, u_a) \\
\tilde{\pi}\big\downarrow & & \pi\big\downarrow & & \big\downarrow\pi \\
(s, x) & \xleftarrow{\ \mathbf{q}_M\ } & (t, x^i) & \xrightarrow[id]{} & (t, x^i).
\end{array} \qquad (3.5)$$

The invariant sections are then

$$u_0 = v(r,x) \quad , \quad u_1 = w(r,x) \quad , \quad u_2 = 0 \quad , \quad u_3 = v(r,x)$$

and the reduced operator is

$$\tilde{\Delta} = (v_{xx} - w_{rx})(du_0 - du_3) \wedge \nu .$$

The infinitesimal linear isotropy representation is

$$\rho_{p_0}(Z) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} .$$

To compute the Palais Condition (2.8) we have

$$[Vert_p\kappa(E)]^{G_x} = \begin{pmatrix} a \\ b \\ 0 \\ a \end{pmatrix} \quad , \quad [Vert_p\kappa(E)^*]^{G_x} = (a,b,0,-a) \quad a,b \in \mathbf{R}$$

and

$$[Vert_p^*(E)]^{G_x} \cap \left([Vert_p(E)]^{G_x}\right)^\perp = \{(a,0,0,-a) , a \in \mathbf{R}\} \neq 0$$

so the principle of symmetric criticality fails. In fact the Lagrangian (3.1) restricted to the invariant sections is easily computed to be

$$\lambda\big|_{\text{Inv}} = 0 .$$

Note however that the Lie algebra cohomology condition (2.4) is satisfied so that the Palais condition is independent of the cohomology condition (2.4).

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